

EXPLICIT SOLUTIONS FOR A NON-CLASSICAL HEAT CONDUCTION PROBLEM FOR A SEMI-INFINITE STRIP WITH A NON-UNIFORM HEAT SOURCE

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Abstract

A non-classical initial and boundary value problem for a non-homogeneous one-dimensional heat equation for a semi-infinite material $x > 0$ with a zero temperature boundary condition at the face $x = 0$ is studied with the aim of finding explicit solutions. It is not a standard heat conduction problem because a heat source $-\Phi(x)F(V(t), t)$ is considered, where Φ and F are real functions and V represents the heat flux at the face $x = 0$.

Explicit solutions independent of the space or temporal variables are given. Solutions with separated variables when the data functions are defined from the solution $X = X(x)$ of a linear initial value problem of second order and the solution $T = T(t)$ of a non-linear (in general) initial value problem of first order which involves the function F , are also given and explicit solutions corresponding to different definitions of the function F are obtained. A solution by an integral representation depending on the heat flux at the boundary $x = 0$ for the case in which $F = F(V, t) = \nu V$, for some $\nu > 0$, is obtained and explicit expressions for the heat flux at the boundary $x = 0$ and for its corresponding solution are calculated when $h = h(x)$ is a potential function and $\Phi = \Phi(x)$ is given by $\Phi(x) = \lambda x$, $\Phi(x) = -\mu \sinh(\lambda x)$ or $\Phi(x) = -\mu \sin(\lambda x)$, for some $\lambda > 0$ and $\mu > 0$.

The limit when the temporal variable t tends to $+\infty$ of each explicit solution obtained in this paper is studied and the "controlling" effects of the source term $-\Phi F$ are analysed by comparing the asymptotic behaviour of each solution with the asymptotic behaviour of the solution to the same problem but in absence of source term.

Finally, a relationship between this problem with another non-classical initial and boundary value problem for the heat equation is established and explicit solutions for this second problem are also obtained.

As a consequence of our study, several problems which can be used as benchmark problems for testing new numerical methods for solving partial differential equations are obtained.

Keywords: Non-classical heat equation, Nonlinear heat conduction problems, Explicit solutions, Volterra integral equations.

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1 Introduction

In this paper we study the following non-classical initial and boundary value problem for a non-homogeneous one-dimensional heat equation (Problem P):

$$u_t(x, t) - u_{xx}(x, t) = -\Phi(x)F(u_x(0, t), t) \quad x > 0, \quad t > 0 \quad (1)$$

$$u(x, 0) = h(x) \quad x > 0 \quad (2)$$

$$u(0, t) = 0 \quad t > 0 \quad (3)$$

with the aim of finding explicit solutions, where $u = u(x, t)$ is the unknown temperature function, defined for $x \geq 0$ and $t \geq 0$, $\Phi = \Phi(x)$, $h = h(x)$ and $F = F(V, t)$ are given functions defined, respectively, for $x > 0$ and $V \in \mathbb{R}$, $t > 0$, and the function h satisfies the following compatibility condition:

$$\lim_{x \rightarrow 0^+} h(x) = 0. \quad (4)$$

This problem is motivated by the regulation of the temperature $u = u(x, t)$ of an isotropic medium, occupying the semi-infinite spatial region $x > 0$, under a non-uniform heat source $-\Phi(x)F(u_x(0, t), t)$ which provides a heater or cooler effect depending on the properties of the function F respect to the heat flux $u_x(0, t)$ at the boundary $x = 0$ [7, 9]. For example, if:

$$\Phi(x) > 0 \quad \text{for } x > 0, \quad \text{and} \quad u_x(0, t)F(u_x(0, t), t) > 0 \quad \forall t > 0,$$

then the source term is a cooler when $u_x(0, t) > 0$ and a heater when $u_x(0, t) < 0$. Some references in this subject are [1, 8, 11, 12, 13, 14, 15, 18, 21, 23, 24, 25]. Problem P for the slab $0 < x < 1$ has been studied in [22]. Recently, free boundary problems (Stefan problems) for the non-classical heat equation have been studied in [3, 4, 5, 6, 10, 17], where some explicit solutions are also given, and a first study of non-classical heat conduction problem for a n -dimensional material has been given in [2]. Numerical schemes for Problem P when a non-homogeneous boundary condition is considered have been studied in [16] and numerical solutions have been given for two particular choices of data function corresponding to problems with known explicit solutions.

The organization of the paper is the following: in Section 2, we give explicit solutions to Problem P. We split this section into three parts. In the first one, we give explicit solutions which are independent of the space variable x or the temporal variable t . In the second part, we find solutions with separated variables when the functions $h = h(x)$ and $\Phi = \Phi(x)$ are proportional to the solution $X = X(x)$ of a linear initial value problem of second order and the function $F = F(V, t)$ is defined from the solution $T = T(t)$ of a non-linear (in general) initial value problem of first order. As a consequence, we give explicit solutions with separated variables corresponding to different definitions of the function F . Finally, in the third part, we find solutions by an integral representation which depends on the heat flux at the boundary $x = 0$ [24] for the case in which F is defined by $F(V, t) = \nu V$, for some $\nu > 0$. Moreover, we find explicit expressions for the heat flux at the boundary $x = 0$ and for its corresponding solution to Problem P, when $h = h(x)$ is a potential function and $\Phi = \Phi(x)$ is given by $\Phi(x) = \lambda x$, $\Phi(x) = -\mu \sinh(\lambda x)$ or $\Phi(x) = -\mu \sin(\lambda x)$, for some $\lambda > 0$ and $\mu > 0$. In Section 3, we deal with the problem of "controlling" solutions of Problem P through the source term $-\Phi(x)F(u_x(0, t), t)$. We compare the asymptotic behaviour of each explicit solution u obtained for the Problem P with the asymptotic behaviour of the solution u_0 of the same problem but in absence of source term, and we obtain conditions for the parameters involved in the definition

of $-\Phi(x)F(V, t)$ under which the asymptotic behaviour of u can be controlled respect to the asymptotic behaviour of u_0 . Finally, in Section 4, we recall the relationship between Problem P with another non-classical initial and boundary value problem for the heat equation [24], given by (Problem \tilde{P}):

$$v_t(x, t) - v_{xx}(x, t) = -\tilde{\Phi}(x)\tilde{F}(v(0, t), t) \quad x > 0, \quad t > 0 \quad (5)$$

$$v(x, 0) = \tilde{h}(x) \quad x > 0 \quad (6)$$

$$v_x(0, t) = \tilde{g}(t) \quad t > 0 \quad (7)$$

and we find explicit solutions to Problem \tilde{P} .

As a consequence of our study, we obtain some particular cases of Problem P and Problem \tilde{P} which can be used as benchmark problems for testing new numerical methods for solving partial differential equations.

2 Explicit solutions for Problem P

2.1 Explicit solutions independents of space or temporal variables

Theorem 2.1.

1. Problem P does not admit any non-trivial solution independent of the space variable x .
2. If:

(a) F is the zero function and h is defined by:

$$h(x) = \eta x, \quad x \geq 0, \quad (8)$$

for some $\eta > 0$,

or

(b) F is a constant function defined by:

$$F(V, t) = \nu, \quad V \in \mathbb{R}, \quad t > 0, \quad (9)$$

for some $\nu \in \mathbb{R} - \{0\}$, and h is a twice differentiable function such that $h(0)$ exists and:

$$h''(x) = \nu\Phi(x), \quad x > 0, \quad (10)$$

then the function u defined by:

$$u(x, t) = h(x), \quad x \geq 0, \quad t \geq 0, \quad (11)$$

is a solution to Problem P independent of the temporal variable t .

Proof.

1. If the Problem P has a solution u independent of the space variable x then:

$$u(x, t) = u(0, t) = 0, \quad x > 0, \quad t > 0 \quad \text{and} \quad u(0, 0) = \lim_{x \rightarrow 0^+} h(x) = 0. \quad (12)$$

Therefore u is the zero function.

2. It is easy to check that the function u given in (11) is a solution to the Problem P given in this item. ■

2.2 Explicit solutions with separated variables

Theorem 2.2. *Let $\lambda, \eta, \delta \in \mathbb{R} - \{0\}$. If Φ, h and F are defined by:*

$$\Phi(x) = \lambda X(x), \quad h(x) = \eta X(x), \quad x > 0 \quad \text{and} \quad F = F(\delta T(t), t), \quad t > 0, \quad (13)$$

where X is given by:

$$X(x) = \begin{cases} \frac{\delta}{\sqrt{\sigma}} \sinh(\sqrt{\sigma}x) & \text{if } \sigma > 0 \\ \frac{\delta}{\sqrt{|\sigma|}} \sin(\sqrt{|\sigma|x}) & \text{if } \sigma < 0 \\ \delta x & \text{if } \sigma = 0 \end{cases}, \quad x \geq 0 \quad (14)$$

and T is the solution of the initial value problem:

$$\dot{T}(t) - \sigma T(t) = -\lambda F(\delta T(t), t), \quad t > 0 \quad (15)$$

$$T(0) = \eta, \quad (16)$$

then the function u given by:

$$u(x, t) = X(x)T(t), \quad x \geq 0, \quad t \geq 0 \quad (17)$$

is a solution with separated variables to Problem P.

Proof. An easy computation shows that the function u given in (17) is a solution to Problem P. ■

Remark 1. The function X given in (14) also can be seen as the solution of a linear initial value problem of second order, in fact X satisfies:

$$X''(x) - \sigma X(x) = 0, \quad x > 0 \quad (18)$$

$$X(0) = 0 \quad (19)$$

$$X'(0) = \delta. \quad (20)$$

Under the hypothesis of the previous theorem, the problem of finding explicit solutions with separated variables to Problem P reduces to solving the initial value problem (15)-(16).

With the spirit of exhibit explicit solutions to Problem P, our next result summarizes explicit solutions to the initial value problem (15)-(16) corresponding to three different definitions of the function F .

Proposition 2.1. *If in Theorem 2.2 we consider:*

1. *Function F defined by:*

$$F(V, t) = \nu V, \quad V \in \mathbb{R}, \quad t > 0, \quad (21)$$

for some $\nu \in \mathbb{R} - \{0\}$, then the function T is given by:

$$T(t) = \eta \exp((\sigma - \lambda \nu \delta)t), \quad t \geq 0. \quad (22)$$

2. Function F defined by:

$$F(V, t) = f_1(t) + f_2(t)V, \quad V \in \mathbb{R}, t > 0, \quad (23)$$

for some $f_1, f_2 \in L_{loc}^1(\mathbb{R}^+)$, then the function T is given by:

$$T(t) = g_1(t) \exp(g_2(t)), \quad t \geq 0, \quad (24)$$

where functions g_1 and g_2 are defined by:

$$g_1(t) = \eta - \lambda \int_0^t f_1(\tau) \exp\left(\lambda \delta \int_0^\tau f_2(\xi) d\xi - \sigma \tau\right) d\tau, \quad t \geq 0 \quad (25)$$

$$g_2(t) = \sigma t - \lambda \delta \int_0^t f_2(\tau) d\tau, \quad t \geq 0. \quad (26)$$

3. Function F defined by:

$$F(V, t) = V^n f(t), \quad V \in \mathbb{R}, t > 0, \quad (27)$$

for some $n < 1$ and some positive function $f \in L_{loc}^1(\mathbb{R}^+)$, and λ, δ and η positive numbers, then the function T is given by:

$$T(t) = g(t) \exp(\sigma t), \quad t \geq 0, \quad (28)$$

where the function g is defined by:

$$g(t) = \left(\eta^{1-n} + \lambda \delta^n (n-1) \int_0^t f(\tau) \exp(\sigma(n-1)\tau) d\tau \right)^{\frac{1}{1-n}}, \quad t \geq 0. \quad (29)$$

Proof. It follows by the application of the integrating factor method to the initial value problem (15)-(16). ■

2.3 Explicit solutions obtained from an integral representation

Our next theorem is a restatement of Theorem 1 in [24] for a particular choice of the function F in Problem P.

Theorem 2.3. *Let:*

1. h a continuously differentiable function in \mathbb{R}^+ such that $h(0)$ exists and there exist positive numbers ϵ, c_0 and c_1 such that:

$$|h(x)| \leq c_0 \exp(c_1 x^{2-\epsilon}), \quad \forall x > 0, \quad (30)$$

2. Φ a locally Hölder continuous function

and

3. F the function defined by:

$$F = F(V, t) = \nu V, \quad V \in \mathbb{R}, t > 0, \quad (31)$$

for some $\nu > 0$.

If there exists a negative monotone decreasing function $f = f(t)$, defined for $t > 0$, such that:

$$\int_{t_1}^{t_2} R(t_2 - \tau) d\tau \geq f(t_2 - t_1), \quad \forall 0 < t_1 < t_2, \quad (32)$$

where R is defined in function of Φ by (40) (see below), and

$$\lim_{t \rightarrow 0^+} f(t) = 0, \quad (33)$$

then the function u defined by:

$$u(x, t) = \int_0^{+\infty} G(x, t, \xi, 0) h(\xi) d\xi - \nu \int_0^t \left(\int_0^{+\infty} G(x, t, \xi, \tau) \Phi(\xi) d\xi \right) V(\tau) d\tau, \quad x \geq 0, t \geq 0 \quad (34)$$

is a solution to Problem P, where G is the Green function:

$$G(x, t, \xi, \tau) = K(x, t, \xi, \tau) - K(-x, t, \xi, \tau), \quad 0 < x, 0 < \xi, 0 < \tau < t, \quad (35)$$

being K the fundamental solution of the one-dimensional heat equation:

$$K(x, t, \xi, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp(-(x-\xi)^2/4(t-\tau)), \quad 0 < x, 0 < \xi, 0 < \tau < t, \quad (36)$$

and the function V , defined by:

$$V(t) = u_x(0, t), \quad t > 0, \quad (37)$$

satisfies the Volterra integral equation:

$$V(t) = V_0(t) - \nu \int_0^t R(t-\tau) V(\tau) d\tau, \quad t > 0, \quad (38)$$

where

$$V_0(t) = \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} \exp(-\xi^2/4t) h'(\xi) d\xi, \quad t > 0, \quad (39)$$

and

$$R(t) = \frac{1}{2\sqrt{\pi} t^{3/2}} \int_0^{+\infty} \xi \exp(-\xi^2/4t) \Phi(\xi) d\xi, \quad t > 0. \quad (40)$$

Remark 2. The interest of the previous theorem is that it enable us to finding an explicit solution $u = u(x, t)$ to Problem P by finding the corresponding heat flux $u_x(0, t)$ at the boundary $x = 0$ as a solution of the integral equation (38).

The remainder of this section will be devoted to the study of Problem P when:

1. F is given as in (31),
2. h is defined by:

$$h(x) = \eta x^m, \quad x > 0, \quad (41)$$

for some $\eta \in \mathbb{R} - \{0\}$ and $m \geq 1$,

and

3. Φ is given by one of the following expressions:

$$\varphi_1(x) = \lambda x, \quad \varphi_2(x) = -\mu \sinh(\lambda x) \quad \text{or} \quad \varphi_3(x) = -\mu \sin(\lambda x), \quad x > 0, \quad (42)$$

for some $\lambda > 0$ and $\mu > 0$.

It is easy to check that for this choice of functions F , h and Φ , Problem P is under the hypothesis of the previous theorem (see Appendix A). Therefore, it has the solution $u = u(x, t)$ given in (34).

Proposition 2.2. *If F , h and $\Phi = \varphi_1$ are defined as in (31), (41) and (42), then the heat flux at the boundary $x = 0$ corresponding to the solution u (see (34)) to Problem P is given by:*

$$u_x(0, t) = \begin{cases} \eta \exp(-\nu \lambda t) & \text{if } m = 1 \\ \frac{c(m-1)}{2} \exp(-\nu \lambda t) \int_0^t \tau^{(m-3)/2} \exp(\nu \lambda \tau) d\tau & \text{if } m > 1 \end{cases}, \quad t > 0. \quad (43)$$

where

$$c = \frac{2^{m-1} m \eta}{\sqrt{\pi}} \Gamma\left(\frac{m}{2}\right), \quad (44)$$

and Γ is the Gamma function, defined by:

$$\Gamma(z) = \int_0^{+\infty} \xi^{z-1} \exp(-\xi) d\xi, \quad z \in \mathbb{R}. \quad (45)$$

Proof. We know from Theorem 2.3 that $u_x(0, t) = V(t)$ satisfies the Volterra integral equation (38), where the function V_0 is given by:

$$V_0(t) = ct^{(m-1)/2}, \quad t > 0. \quad (46)$$

Then, $V(t)$ is given by (see [19]):

$$V(t) = \frac{c(m-1)}{2} \int_0^t \tau^{(m-3)/2} r(\tau) d\tau, \quad t > 0, \quad (47)$$

where r satisfies the integral equation:

$$r(t) = 1 - \nu \lambda \int_0^t r(\tau) d\tau, \quad t > 0, \quad (48)$$

whose solution is given by:

$$r(t) = \exp(-\nu \lambda t), \quad t > 0. \quad (49)$$

By replacing (49) in (47), we obtain (43). ■

Corollary 2.1. *If in Proposition 2.2 we consider m an odd number given by $m = 2p + 1$ with $p \in \mathbb{N}$, then we have:*

$$u_x(0, t) = p_{1,m}(t) - c_{1,m} \exp(-\nu \lambda t), \quad t > 0, \quad (50)$$

where $c_{1,m}$ is given by:

$$c_{1,m} = (-1)^{p-1} \frac{cp!}{(\nu\lambda)^p}, \quad (51)$$

being c the constant given in (44), and $p_{1,m}(x)$ is the polynomial defined by:

$$p_{1,m}(t) = \begin{cases} c_{1,3} & \text{if } m = 3 \\ -c_{1,5}(\nu\lambda t - 1) & \text{if } m = 5 \\ c_{1,m} \left(\sum_{k=1}^{p-1} \frac{(-\nu\lambda)^k}{k!} t^k + (-1)^{p-1} \right) & \text{if } m \geq 7 \end{cases}, \quad t > 0. \quad (52)$$

Proof. It follows by solving the integral in the expression of $u_x(0, t)$ given in (43). We do not reproduce these calculations here, but only remark the utility of the identity:

$$\int_0^t \tau^n \exp(a\tau) d\tau = \frac{n!}{a} \exp(at) \left(\sum_{k=0}^{n-1} \frac{(-1)^k t^{n-k}}{(n-k)! a^k} + \frac{(-1)^n}{a^n} + \frac{(-1)^{n+1}}{a^n} \exp(-at) \right), \quad (53)$$

$t > 0, n \in \mathbb{N}, n \geq 2, a \in \mathbb{R}$

when $m \geq 7$. ■

Last corollary enables us to obtain the asymptotic behaviour of the heat flux $u_x(0, t)$ at the face $x = 0$ when t tends to $+\infty$, for an odd number m . Next result is related to this topic. We do not reproduce here the computations involved in its proof, which follows by taking the limit when t tends to $+\infty$ in the expression of $u_x(0, t)$ given in Corollary 2.1.

Corollary 2.2. *If F , h and $\Phi = \varphi_1$ are defined as in (31), (41) and (42), where m is an odd number, and u is the solution to Problem P, given in (34), then:*

1. if $m = 1$, we have:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = 0, \quad (54)$$

2. if $m = 3$, we have:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \frac{6\eta}{\nu\lambda}, \quad (55)$$

3. if $m \geq 5$, we have:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \eta < 0 \\ +\infty & \text{if } \eta > 0 \end{cases}. \quad (56)$$

The main idea in the proof of Proposition 2.2 was to find a solution for the integral equation (38) by finding a solution of another integral equation, which was easier to solve. In a more general way, we know that if V satisfies the Volterra integral equation (38), with V_0 an infinitely differentiable function, then $V(t)$ can be written as (see [19]):

$$V(t) = V_0(0)r(t) + \int_0^t V_0'(t - \tau)r(\tau)d\tau, \quad t > 0, \quad (57)$$

where r satisfies the integral equation:

$$r(t) = 1 - \nu \int_0^t R(t - \tau)r(\tau)d\tau, \quad t > 0, \quad (58)$$

and R is given in (40). But this last integral equation is not always easy to solve. Nevertheless, in several cases we can find an explicit solution for the equation (58) by a formal application of the Laplace transform to their both sides. This is the way which led us to the expressions of $u_x(0, t)$ when $\Phi = \varphi_2$ or $\Phi = \varphi_3$, given in Propositions 2.3 and 2.4.

Proposition 2.3. *Let F , h and $\Phi = \varphi_2$ defined as in (31), (41) and (42), and $\sigma = \lambda + \nu\mu$. Then the heat flux at the boundary $x = 0$ corresponding to the solution u (see (34)) of the Problem P is given by:*

1. *If $\sigma \neq 0$ then:*

$$u_x(0, t) = \begin{cases} \frac{\lambda}{\sigma} (\lambda + \nu\mu \exp(\lambda\sigma t)) & \text{if } m = 1 \\ \frac{c\lambda}{\sigma} t^{(m-1)/2} + \frac{c(m-1)\nu\mu}{2\sigma} \exp(\lambda\sigma t) \int_0^t \tau^{(m-3)/2} \exp(-\lambda\sigma\tau) d\tau & \text{if } m > 1 \end{cases}, \quad t > 0. \quad (59)$$

2. *If $\sigma = 0$ then:*

$$u_x(0, t) = \begin{cases} \eta(1 - \lambda^2 t) & \text{if } m = 1 \\ ct^{(m-1)/2} - \frac{2c\lambda^2}{m+1} t^{(m+1)/2} & \text{if } m > 1 \end{cases}, \quad t > 0. \quad (60)$$

Proof. An easy computation shows that the expressions given in (59) and (60) satisfy the integral equation (38). Therefore, they correspond to the heat flux $u_x(0, t)$ at the boundary $x = 0$ for the solution u of the Problem P given in (34). ■

Corollary 2.3. *If in Proposition 2.3 we consider $\sigma \neq 0$ and m an odd number given by $m = 2p + 1$ with $p \in \mathbb{N}$, then we have:*

$$u_x(0, t) = p_{2,m}(t) + c_{2,m} \exp(\lambda\sigma t), \quad t > 0, \quad (61)$$

where $c_{2,m}$ is given by:

$$c_{2,m} = \frac{c\nu\mu p!}{\sigma(\lambda\sigma)^p}, \quad (62)$$

being c the constant given in (44), and $p_{2,m}(x)$ is the polynomial defined by:

$$p_{2,m}(t) = \begin{cases} c_{2,3} \left(\frac{\lambda^2 \sigma}{\nu\mu} t - 1 \right) & \text{if } m = 3 \\ c_{2,5} \left(\frac{\lambda^3 \sigma^2}{2\nu\mu} t^2 - \lambda\sigma t - 1 \right) & \text{if } m = 5 \\ \frac{c\lambda}{\sigma} t^p - c_{2,m} \left(\sum_{k=1}^{p-1} \frac{(\lambda\sigma)^k}{k!} t^k + 1 \right) & \text{if } m \geq 7 \end{cases}, \quad t > 0. \quad (63)$$

Proof. It follows by solving the integral in the expression of u given in (59) and the use of the identity (53). ■

Corollary 2.4. *Let F , h and $\Phi = \varphi_2$ defined as in (31), (41) and (42), with m and odd number, and $\sigma = \lambda + \nu\mu$. If u is the solution of the Problem P, given in (34), then:*

1. *If $\sigma \neq 0$ then:*

(a) if $m = 1$, we have:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \sigma > 0, \eta < 0 \\ +\infty & \text{if } \sigma > 0, \eta > 0 \\ \frac{\eta\lambda}{\sigma} & \text{if } \sigma < 0 \end{cases}, \quad (64)$$

(b) if $m \geq 3$, we have:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \sigma\eta < 0 \\ +\infty & \text{if } \sigma\eta > 0 \end{cases}. \quad (65)$$

2. If $\sigma = 0$ then:

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \eta > 0 \\ +\infty & \text{if } \eta < 0 \end{cases}. \quad (66)$$

Proposition 2.4. Let F , h and $\Phi = \varphi_3$ defined as in (31), (41) and (42), and $\delta = \lambda - \nu\mu$. Then the heat flux at the boundary $x = 0$ corresponding to the solution u (see (34)) of the Problem P is given by:

1. if $\delta \neq 0$ then:

$$u_x(0, t) = \begin{cases} \frac{\eta}{\delta} (\lambda - \nu\mu \exp(-\lambda\delta t)) & \text{if } m = 1 \\ \frac{c\lambda}{\delta} t^{(m-1)/2} - \frac{c(m-1)\nu\mu}{2\delta} \exp(-\lambda\delta t) \int_0^t \tau^{(m-3)/2} \exp(\lambda\delta\tau) d\tau & \text{if } m > 1 \end{cases}, \quad t > 0. \quad (67)$$

2. If $\delta = 0$ then:

$$u_x(0, t) = \begin{cases} \eta(1 + \lambda^2 t) & \text{if } m = 1 \\ ct^{(m-1)/2} + \frac{2c\lambda^2}{m+1} t^{(m+1)/2} & \text{if } m > 1 \end{cases}, \quad t > 0. \quad (68)$$

Proof. The proof of (67) and (68) follows by replacing λ^2 by $-\lambda^2$ and σ by δ in the proof of Proposition 2.3. ■

Corollary 2.5. If in Proposition 2.3 we consider $\delta \neq 0$ and m an odd number given by $m = 2p + 1$ with $p \in \mathbb{N}$, then we have:

$$u_x(0, t) = p_{3,m}(c) + c_{3,m} \exp(-\lambda\delta t), \quad t > 0, \quad (69)$$

where $c_{3,m}$ is given by:

$$c_{3,m} = (-1)^{p-1} \frac{c\nu\mu p!}{\delta(\lambda\delta)^p}, \quad (70)$$

being c the constant given in (44), and $p_{3,m}(x)$ is the polynomial defined by:

$$p_{3,m}(t) = \begin{cases} c_{3,3} \left(\frac{\lambda^2\delta}{\nu\mu} t - 1 \right) & \text{if } m = 3 \\ -c_{3,5} \left(\frac{\lambda^3\delta^2}{2\nu\mu} t^2 - \lambda\sigma t + 1 \right) & \text{if } m = 5 \\ \frac{c\lambda}{\delta} t^p - c_{3,m} \left(\sum_{k=1}^{p-1} \frac{(-\lambda\delta)^k}{k!} t^k + 1 \right) & \text{if } m \geq 7 \end{cases}, \quad t > 0. \quad (71)$$

Proof. It follows by solving the corresponding integral in the expression (67) and the use of the identity (53). ■

Corollary 2.6. *Let F , h and $\Phi = \varphi_3$ defined as in (31), (41) and (42), with m and odd number, and $\delta = \lambda - \nu\mu$. If u is the solution of the Problem P, given in (34), then:*

1. *If $\delta \neq 0$ then:*

(a) *if $m = 1$, we have:*

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \delta < 0, \quad \eta < 0 \\ +\infty & \text{if } \delta < 0, \quad \eta > 0 \\ \frac{\eta\lambda}{\delta} & \text{if } \delta > 0 \end{cases}, \quad (72)$$

(b) *if $m = 3$ or $m = 5$, we have:*

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \eta < 0 \\ +\infty & \text{if } \eta > 0 \end{cases} \quad (73)$$

and

(c) *if $m \geq 7$, we have:*

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \delta\eta < 0 \\ +\infty & \text{if } \delta\eta > 0 \end{cases}. \quad (74)$$

2. *If $\delta = 0$ then:*

$$\lim_{t \rightarrow +\infty} u_x(0, t) = \begin{cases} -\infty & \text{if } \eta < 0 \\ +\infty & \text{if } \eta > 0 \end{cases}. \quad (75)$$

Next result is related to the behaviour of the heat flux $u_x(0, t)$ at the face $x = 0$ when t tends to 0^+ , and shows that it is independent on the choice of Φ as any of the functions given in (42).

Corollary 2.7. *If F , h and Φ are given as in (31), (41) and any of the expressions in (42), respectively, then:*

$$\lim_{t \rightarrow 0^+} u_x(0, t) = \begin{cases} \eta & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}, \quad (76)$$

where u is the solution of the Problem P given in (34).

Proof. It follows straightforward by computing the limit for the expression of $u_x(0, t)$ given in Proposition 2.2, 2.3 or 2.4, according the definition of Φ . ■

We end this section by giving explicit solutions to each Problem P. The proofs of the three following propositions follow from Theorem 2.3 and Corollary 2.1, 2.3 or 2.5, according to the definition of Φ (see Appendix B).

Proposition 2.5. *If F , h and $\Phi = \varphi_1$ are defined as in (31), (41) and (42), where m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by:*

$$u(x, t) = u_0(x, t) - \nu\Phi(x) \int_0^t V(\tau) d\tau, \quad x \geq 0, t \geq 0 \quad (77)$$

is a solution to Problem P, where u_0 is defined by:

$$u_0(x, t) = \frac{\eta}{\sqrt{\pi}} \sum_{k=0}^p \binom{m}{2k} \Gamma\left(\frac{2k+1}{2}\right) (4t)^k x^{m-2k}, \quad x \geq 0, t \geq 0, \quad (78)$$

and $V(t) = u_x(0, t)$ is given by (50).

Remark 3. If $m = 1$, polynomial $p_{1,m}(x)$ is defined by $p_{1,m}(x) = 0$, $x > 0$.

Proposition 2.6. If F , h and $\Phi = \varphi_2$ are defined as in (31), (41) and (42), where $\sigma \neq 0$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by:

$$u(x, t) = u_0(x, t) - \nu \Phi(x) \exp(\lambda^2 t) \int_0^t V(\tau) \exp(-\lambda^2 \tau) d\tau, \quad x \geq 0, t \geq 0 \quad (79)$$

is a solution to Problem P, where u_0 and $V(t) = u_x(0, t)$ are given by (78) and (61).

Remark 4. If $m = 1$, polynomial $p_{2,m}(x)$ is defined by $p_{2,m}(x) = \frac{\nu\lambda}{\sigma}$, $x > 0$.

Proposition 2.7. If F , h and $\Phi = \varphi_3$ are defined as in (31), (41) and (42), where $\delta \neq 0$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by:

$$u(x, t) = u_0(x, t) - \nu \Phi(x) \exp(-\lambda^2 t) \int_0^t V(\tau) \exp(\lambda^2 \tau) d\tau, \quad x \geq 0, t \geq 0 \quad (80)$$

is a solution to Problem P, where u_0 and $V(t) = u_x(0, t)$ are given by (78) and (69).

Remark 5. If $m = 1$, polynomial $p_{3,m}(x)$ is defined by $p_{3,m}(x) = \frac{\nu\lambda}{\delta}$, $x > 0$.

3 The controlling problem

This section is devoted to study the effects introduced by the source term $-\Phi F$ in the asymptotic behaviour of the solution u to each Problem P considered in this paper. We will carry out our study by comparing the asymptotic behaviour of u with the asymptotic behaviour of the solution u_0 to Problem P in the absence of control (Problem P₀):

$$u_t(x, t) - u_{xx}(x, t) = 0 \quad x > 0, \quad t > 0 \quad (81)$$

$$u(x, 0) = h(x) \quad x > 0 \quad (82)$$

$$u(0, t) = 0 \quad t > 0 \quad (83)$$

This kind of analysis enable us to control Problem P by its source term.

The study of controlling Problem P by its source term has been done in [1] when Φ is identically equal to 1, $F = F(V)$ is a differentiable function of one real variable which satisfies:

1. $VF(V) \geq 0$, $\forall V \in \mathbb{R}$,
2. $F(0) = 0$,
3. F is convex in $(0, +\infty)$,
4. $\lim_{V \rightarrow +\infty} F'(V) = \kappa > 0$,

and h is a non-negative, continuous and bounded function. They proved that under these hypothesis, both u and u_0 converge to 0 when t tends to $+\infty$ and the control term F has a stabilizing effect because $\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = 0$, that is, u converge faster to 0 than u_0 . None of the cases studied in the previous sections fulfil the hypothesis for Φ , F and h established in [1].

With the aim of supplementing the results given in [1], we will carry out our analysis under conditions which lead us to functions F depending on only one real variable, that is $F = F(V)$.

Next Theorems 3.1, 3.2 and 3.3 are respectively related with the results obtained in Sections 2.1, 2.2 and 2.3.

Remark 6. For all Problems P studied in this paper, Problem P_0 has the solution u_0 defined by [7]:

$$u_0(x, t) = \int_0^{+\infty} G(x, t, \xi, 0) h(\xi) d\xi, \quad x \geq 0, t \geq 0, \quad (84)$$

where G is the Green function defined in (35).

Theorem 3.1. *Let Φ identically equal to 1, F a constant function defined by:*

$$F(V) = \nu, \quad V \in \mathbb{R}, \quad (85)$$

for some $\nu \in \mathbb{R} - \{0\}$, and h a quadratic function defined by:

$$h(x) = \frac{\nu}{2}x^2 + ax, \quad x \geq 0, \quad (86)$$

for some $a \in \mathbb{R}$.

For the solution u_0 to Problem P_0 given in (84), we have:

$$\lim_{t \rightarrow +\infty} u_0(x, t) = \infty, \quad \forall x > 0. \quad (87)$$

Furthermore, there exists a solution u to Problem P such that:

$$\lim_{t \rightarrow +\infty} u(x, t) = h(x), \quad \forall x > 0. \quad (88)$$

Proof. By computing the integral in (84) for the function h given in (86), we have that the solution u_0 to Problem P_0 given in (84) is defined by:

$$u_0(x, t) = \left(\frac{\nu}{2}x^2 + \nu t\right) \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + \frac{\nu}{\sqrt{\pi}}x\sqrt{t} \exp\left(\frac{-x^2}{4t}\right) + ax, \quad x > 0, t > 0. \quad (89)$$

By taking the limit when t tends to $+\infty$, we have (87).

Since functions Φ , F and h are under the hypothesis of Theorem 2.1, we know that the function u given by:

$$u(x, t) = h(x), \quad x \geq 0, t \geq 0, \quad (90)$$

is a solution to Problem P, which satisfies (88). ■

Theorem 3.2. *Let Φ , h and F defined by:*

$$\Phi(x) = \lambda X(x), \quad h(x) = \eta X(x), \quad x > 0 \quad \text{and} \quad F = F(\delta T(t)), \quad t > 0, \quad (91)$$

where X is the function given by (14):

$$X(x) = \begin{cases} \frac{\delta}{\sqrt{\sigma}} \sinh(\sqrt{\sigma}x) & \text{if } \sigma > 0 \\ \frac{\delta}{\sqrt{|\sigma|}} \sin(\sqrt{|\sigma|x}) & \text{if } \sigma < 0 \\ \delta x & \text{if } \sigma = 0 \end{cases}, \quad x > 0$$

and T is the solution of the initial value problem (15)-(16):

$$\begin{aligned} \dot{T}(t) - \sigma T(t) &= -\lambda F(\delta T(t), t), \quad t > 0 \\ T(0) &= \eta, \end{aligned}$$

with $\lambda, \eta, \delta \in \mathbb{R} - \{0\}$.

For the solution u_0 to Problem P_0 given in (84), we have:

$$\lim_{t \rightarrow +\infty} u_0(x, t) = \begin{cases} h(x) & \text{if } \sigma = 0 \\ \infty & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma < 0 \end{cases}, \quad \forall x > 0. \quad (92)$$

Furthermore:

1. If F is defined by:

$$F(V) = \nu V, \quad V \in \mathbb{R}, \quad (93)$$

for some $\nu \in \mathbb{R} - \{0\}$, then there exists a solution u to Problem P which satisfies:

$$\lim_{t \rightarrow +\infty} u(x, t) = \begin{cases} h(x) & \text{if } \gamma = \sigma \\ \infty & \text{if } \gamma < \sigma \\ 0 & \text{if } \gamma > \sigma \end{cases}, \quad \begin{aligned} &\forall x > 0 \text{ if } \gamma \geq \sigma \\ &\forall x > 0 / h(x) \neq 0 \text{ if } \gamma < \sigma \end{aligned}, \quad (94)$$

being $\gamma = \lambda \nu \delta$.

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = \begin{cases} \infty & \text{if } \gamma < 0 \\ 0 & \text{if } \gamma > 0 \end{cases}, \quad \begin{aligned} &\forall x > 0 \text{ if } \sigma \geq 0 \\ &\forall x > 0 / h(x) \neq 0 \text{ if } \sigma < 0 \end{aligned}. \quad (95)$$

2. If F is defined by:

$$F(V) = \nu V^n, \quad V \in \mathbb{R}, \quad (96)$$

for some $\nu > 0$ and $n < 1$, and we consider $\lambda > 0$, $\eta > 0$ and $\delta > 0$, then there exists a solution u to Problem P which satisfies:

$$\lim_{t \rightarrow +\infty} u(x, t) = \begin{cases} 0 & \text{if } \sigma < 0 \vee 0 < n < 1 \\ \theta_1 h(x) & \text{if } \sigma < 0 \wedge n = 0 \\ \infty & \text{if } \sigma \geq 0 \vee (\sigma < 0 \wedge n < 0) \end{cases}, \quad \begin{aligned} &\forall x > 0 \text{ if } \sigma < 0 \wedge 0 \leq n < 1 \\ &\forall x > 0 / h(x) \neq 0 \text{ if } \\ &\sigma \geq 0 \vee (n < 0 \wedge \sigma < 0) \end{aligned}, \quad (97)$$

where $\theta_1 = \frac{\lambda \mu}{\sigma \eta}$.

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = \begin{cases} \infty & \text{if } \sigma \leq 0 \\ 1 - \frac{\lambda \mu \delta^n}{\sigma \eta^{1-n}} & \text{if } \sigma > 0 \end{cases}, \quad \begin{aligned} &\forall x > 0 \text{ if } \sigma \geq 0 \\ &\forall x > 0 / h(x) \neq 0 \text{ if } \sigma < 0 \end{aligned}. \quad (98)$$

Proof. By computing the integral in (84) for the function h given in (91), we obtain that the solution u_0 to Problem P_0 given in (84) is defined by:

$$u_0(x, t) = h(x) \exp(|\sigma|t), \quad \forall x > 0, t > 0. \quad (99)$$

By taking the limit when t tends to $+\infty$, we have (92).

1. Since Φ , h and F are under the hypothesis of Corollary 2.1, we know that the function u given by:

$$u(x, t) = \eta X(x) \exp((\sigma - \gamma)t), \quad \forall x > 0, t > 0 \quad (100)$$

is a solution to Problem P , which satisfies (94). Finally, the proof of (95) follows straightforward by computing the limit from the explicit expressions of u_0 and u given in (99) and (100).

2. It follows in the same manner that the proof of the previous item. ■

We see from the previous theorem that we can control the Problem P through the parameters involved in the definition of the source term $-\Phi F$. When $F(V) = \nu V$, we can increase ($\gamma < 0 < \sigma$) or decrease ($0 < \gamma < \sigma$) the velocity of convergence to ∞ for u respect to the velocity of convergence for u_0 . We also can stabilize the problem by doing u tending to a constant value ($0 < \sigma \leq \gamma$) when u_0 is going to ∞ . When $F(V) = \nu V^n$, we can decrease ($\sigma > 0$ and $1 = \frac{\lambda \mu \delta^n}{\sigma \eta^{1-n}}$) or maintain ($\sigma > 0$ and $1 \neq \frac{\lambda \mu \delta^n}{\sigma \eta^{1-n}}$) the velocity of convergence to ∞ for u respect to the velocity of convergence for u_0 . We also can decrease the velocity of convergence to 0 for u respect to the velocity of convergence for u_0 ($\sigma < 0$).

Theorem 3.3. *Let Φ defined by one of the expressions given in (42):*

$$\varphi_1(x) = \lambda x, \quad \varphi_2(x) = -\mu \sinh(\lambda x) \quad \text{or} \quad \varphi_3(x) = -\mu \sin(\lambda x), \quad x > 0,$$

where $\lambda > 0$ and $\mu > 0$, F defined by:

$$F = F(V) = \nu V, \quad V \in \mathbb{R}, \quad (101)$$

for some $\nu > 0$ and h defined as in (41):

$$h(x) = \eta x^m, \quad x > 0,$$

where $\eta \in \mathbb{R} - \{0\}$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$.

For the solution u_0 to Problem P_0 given in (84), we have:

$$\lim_{t \rightarrow +\infty} u_0(x, t) = \begin{cases} h(x) & \text{if } m = 1 \\ \infty & \text{if } m > 1 \end{cases}, \quad \forall x > 0. \quad (102)$$

Furthermore:

1. If $\Phi = \varphi_1$, then there exists a solution u to Problem P which satisfies:

$$\lim_{t \rightarrow +\infty} u(x, t) = \begin{cases} 0 & \text{if } m = 1 \\ \infty & \text{if } m > 1 \end{cases}, \quad \forall x > 0. \quad (103)$$

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = \begin{cases} 0 & \text{if } m = 1 \\ r(x) & \text{if } m > 1 \end{cases}, \quad \forall x > 0, \quad (104)$$

being $r(x)$ is a rational function in the variable x .

2. If $\Phi = \varphi_2$, then there exists a solution u to Problem P which satisfies:

$$\lim_{t \rightarrow +\infty} u(x, t) = \infty, \quad \forall x > 0. \quad (105)$$

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = \infty, \quad \forall x > 0. \quad (106)$$

3. If $\Phi = \varphi_3$, then there exists a solution u to Problem P which satisfies:

$$\lim_{t \rightarrow +\infty} u(x, t) = \infty, \quad \forall x > 0. \quad (107)$$

Therefore,

$$\lim_{t \rightarrow +\infty} \frac{u(x, t)}{u_0(x, t)} = \begin{cases} r(x) & \text{if } \delta > 0 \\ \infty & \text{if } \delta \leq 0 \end{cases}, \quad \forall x > 0, \quad (108)$$

where $r(x)$ is a rational function in the variable x .

Proof. It follows in the same manner that the proofs of Theorems 3.1 and 3.2. ■

From the previous theorem, we see again that there exist several cases where we can control the Problem P through the source term $-\Phi F$.

4 Explicit solutions for Problem \tilde{P}

The following theorem states a relationship between the Problem P and the Problem \tilde{P} given in (5)-(7), and it was proved in [24].

Theorem 4.1. *If u is a solution to Problem P where h and Φ are differentiable functions in \mathbb{R}^+ , then the function v defined by:*

$$v(x, t) = u_x(x, t), \quad x \geq 0, t \geq 0 \quad (109)$$

is a solution to Problem \tilde{P} when \tilde{F} , $\tilde{\Phi}$, \tilde{h} and \tilde{g} are defined by:

$$\begin{aligned} \tilde{F}(V, t) &= F(V, t), \quad V > 0, t > 0, & \tilde{g}(t) &= \Phi(0)F(u_x(0, t), t), \quad t > 0, \\ \tilde{\Phi}(x) &= \Phi'(x), \quad x > 0, & \tilde{h}(x) &= h'(x), \quad x > 0. \end{aligned} \quad (110)$$

We end this section by giving explicit solutions for some particular cases of Problem \tilde{P} .

Proposition 4.1. *Let \tilde{g} the zero function and:*

1. (a) \tilde{F} the zero function and \tilde{h} a constant function, or
- (b) \tilde{F} a constant function defined by:

$$\tilde{F}(V, t) = k, \quad V \in \mathbb{R}, t > 0, \quad (111)$$

for some $k \in \mathbb{R} - \{0\}$, $\tilde{\Phi}$ a locally integrable function in \mathbb{R}^+ and \tilde{h} a differentiable function such that:

$$\tilde{h}(x) = k \int_0^x \tilde{\Phi}(\xi) d\xi, \quad x > 0. \quad (112)$$

Then the function v defined by:

$$v(x, t) = \tilde{h}(x), \quad x \geq 0, t \geq 0 \quad (113)$$

is a solution to Problem \tilde{P} independent of the temporal variable t .

2. \tilde{F} given by (21), (23) or (27), that is:

$$\begin{aligned} F(V, t) &= \nu V, \quad V \in \mathbb{R}, t > 0, \text{ with } \nu \in \mathbb{R} - \{0\}, \\ F(V, t) &= f_1(t) + f_2(t)V, \quad V \in \mathbb{R}, t > 0, \text{ with } f_1, f_2 \in L^1_{loc}(\mathbb{R}^+), \text{ or} \\ F(V, t) &= V^n f(t), \quad V \in \mathbb{R}, t > 0, \text{ with } n < 1, f \in L^1_{loc}(\mathbb{R}^+), f > 0 \text{ and } \lambda, \delta, \eta > 0, \end{aligned}$$

and \tilde{h} and $\tilde{\Phi}$ defined by:

$$\tilde{h}(x) = \eta \tilde{X}(x) \quad \text{and} \quad \tilde{\Phi}(x) = \lambda \tilde{X}(x), \quad t > 0, \quad (114)$$

where \tilde{X} is given by:

$$\tilde{X}(x) = \begin{cases} \delta \cosh(\sqrt{\sigma})x & \text{if } \sigma > 0 \\ \delta \cos(\sqrt{|\sigma|})x & \text{if } \sigma < 0 \\ \delta & \text{if } \sigma = 0 \end{cases}, \quad x > 0, \quad (115)$$

$\lambda, \eta, \delta \in \mathbb{R} - \{0\}$.

Then the function v defined by:

$$v(x, t) = \tilde{X}(x) \tilde{T}(t), \quad x \geq 0, t \geq 0 \quad (116)$$

is a solution with separated variables to Problem \tilde{P} , where \tilde{T} the solution of the initial value problem (15)-(16).

3. \tilde{F} defined as in (31):

$$F = F(V, t) = \nu V, \quad V \in \mathbb{R}, t > 0,$$

for some $\nu > 0$, \tilde{h} defined as:

$$\tilde{h}(x) = \tilde{\eta} x^l, \quad x > 0, \quad (117)$$

for some $\tilde{\eta} \in \mathbb{R} - \{0\}$ and $l \geq 0$, and $\tilde{\Phi}$ given by one of the following expressions:

$$\tilde{\varphi}_1(x) = \tilde{\lambda} x, \quad \varphi_2(x) = -\tilde{\mu} \cosh(\tilde{\lambda} x) \quad \text{or} \quad \varphi_3(x) = -\tilde{\mu} \cos(\tilde{\lambda} x), \quad x > 0, \quad (118)$$

for some $\tilde{\lambda} > 0$ and $\tilde{\nu} > 0$. Then the function v defined by:

$$v(x, t) = u_x(x, t), \quad x \geq 0, t \geq 0 \quad (119)$$

is a solution to Problem \tilde{P} , where u is given by (77) if $\tilde{\Phi} = \tilde{\varphi}_1$, by (79) if $\tilde{\Phi} = \tilde{\varphi}_2$ or by (80) if $\tilde{\Phi} = \tilde{\varphi}_3$.

Proof. It follows from the previous theorem and the explicit solutions to Problem P obtained in Section 2. ■

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Appendices

A Problem P is under the hypothesis of Theorem 2.3

Let a Problem P with F and h given as in (31) and (41), respectively, and Φ defined by any of the expressions φ_1 , φ_2 or φ_3 given in (42).

1. It is clear that h is a continuously differentiable function such that $h(0)$ exists. We also have:

$$|h(x)| = |\eta|x^m \leq |\eta|(x+1)^m \leq |\eta|\exp(mx), \quad \forall x > 0. \quad (120)$$

Then the inequality (30) holds with $\epsilon = 1$, $c_1 = m$ and $c_0 = |\eta|$.

2. It is easy to check that each of the functions φ given in (42) is uniformly Hölder continuous, with Hölder exponent $\alpha = 1$, on any compact set $K \subset \mathbb{R}$.
3. Hypothesis 3 holds because of the definition of the Problem P.

Furthermore, we have:

i. If $\Phi = \varphi_1$, then:

$$R(t) = \lambda, \quad t > 0. \quad (121)$$

Then, the inequality (32) holds with a function f defined by:

$$f(t) = -\lambda t, \quad t > 0. \quad (122)$$

ii. If $\Phi = \varphi_2$, then:

$$R(t) = -\lambda \mu \exp(\lambda^2 t), \quad t > 0. \quad (123)$$

Then the inequality (32) holds with a function f defined by:

$$f(t) = -\frac{\mu}{\lambda} (\exp(\lambda^2 t) - 1), \quad t > 0. \quad (124)$$

iii. If $\Phi = \varphi_3$, then:

$$R(t) = -\lambda \mu \exp(-\lambda^2 t), \quad t > 0. \quad (125)$$

Then the inequality (32) holds with a function f defined by:

$$f(t) = -\frac{\mu}{\lambda} (1 - \exp(-\lambda^2 t)), \quad t > 0. \quad (126)$$

B Proof of Propositions 2.5, 2.6 and 2.7

Let a Problem P with F and h given as in (31) and (41), respectively, and Φ given by any of the expressions in (42).

Computation of $\int_0^{+\infty} G(x, t, \xi, 0) h(\xi) d\xi$

By the definitions of the functions G and h given in (35) and (41), respectively, we have:

$$\int_0^{+\infty} G(x, t, \xi, 0) h(\xi) d\xi = \frac{\eta}{2\sqrt{\pi t}} \int_0^{+\infty} (\exp(-(x - \xi)^2/4t) - \exp(-(x + \xi)^2/4t)) \xi^m d\xi. \quad (127)$$

We first compute $\int_0^{+\infty} \exp(-(x - \xi)^2/4t) \xi^m d\xi$.

By doing the substitution $\zeta = (x - \xi)/2\sqrt{t}$, we have:

$$\begin{aligned} \int_0^{+\infty} \exp(-(x - \xi)^2/4t) \xi^m d\xi &= 2\sqrt{t} \int_{-\infty}^{x/2\sqrt{t}} \exp(-\zeta^2) \left(x - 2\sqrt{t}\zeta\right)^m d\zeta \\ &= 2\sqrt{t} \sum_{k=0}^m \binom{m}{k} (-2\sqrt{t})^k x^{m-k} \int_{-\infty}^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta. \end{aligned} \quad (128)$$

Since:

$$\begin{aligned}
\int_{-\infty}^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta &= \int_{-\infty}^0 \exp(-\zeta^2) \zeta^k d\zeta + \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta \\
&= (-1)^k \int_0^{+\infty} \exp(-\sigma^2) \sigma^k d\sigma + \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta \\
&= \frac{(-1)^k}{2} \Gamma\left(\frac{k+1}{2}\right) + \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta,
\end{aligned} \tag{129}$$

then we have:

$$\begin{aligned}
\int_0^{+\infty} \exp(-(x-\xi)^2/4t) \xi^m d\xi &= \sqrt{t} \sum_{k=0}^m \binom{m}{k} (2\sqrt{t})^k x^{m-k} \Gamma\left(\frac{k+1}{2}\right) + \\
&\quad 2\sqrt{t} \sum_{k=0}^m \binom{m}{k} (-2\sqrt{t})^k x^{m-k} \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta.
\end{aligned} \tag{130}$$

By similar calculations, we have:

$$\begin{aligned}
\int_0^{+\infty} \exp(-(x+\xi)^2/4t) \xi^m d\xi &= \sqrt{t} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} (2\sqrt{t})^k x^{m-k} \Gamma\left(\frac{k+1}{2}\right) + \\
&\quad 2\sqrt{t} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} (-2\sqrt{t})^k x^{m-k} \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta.
\end{aligned} \tag{131}$$

Therefore, we have:

$$\begin{aligned}
\int_0^{+\infty} G(x, t, \xi, 0) h(\xi) d\xi &= \frac{\eta}{2\sqrt{\pi t}} \left(\sqrt{t} \sum_{k=0}^m (1 - (-1)^{m-k}) \binom{m}{k} (2\sqrt{t})^k x^{m-k} \Gamma\left(\frac{k+1}{2}\right) + \right. \\
&\quad \left. 2\sqrt{t} \sum_{k=0}^m (1 - (-1)^{m+1}) \binom{m}{k} (-2\sqrt{t})^k x^{m-k} \int_0^{x/2\sqrt{t}} \exp(-\zeta^2) \zeta^k d\zeta \right) \\
&= \frac{\eta}{2\sqrt{\pi}} \sum_{k=0}^m (1 - (-1)^{k+1}) \binom{m}{k} (2\sqrt{t})^k x^{m-k} \Gamma\left(\frac{k+1}{2}\right) \\
&= \frac{\eta}{\sqrt{\pi}} \sum_{k=0}^p \binom{m}{2k} (4t)^k x^{m-2k} \Gamma\left(\frac{2k+1}{2}\right), \quad x > 0.
\end{aligned} \tag{132}$$

Computation of $\int_0^{+\infty} G(x, t, \xi, \tau) \Phi(\xi) d\xi$

1. By the definitions of the functions G and $\Phi = \varphi_1$ given in (35) and (42), respectively, we have:

$$\begin{aligned}
\int_0^{+\infty} G(x, t, \xi, \tau) \varphi_1(\xi) d\xi &= \frac{\lambda}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} (\exp(-(x-\xi)^2/4(t-\tau)) - \\
&\quad \exp(-(x+\xi)^2/4(t-\tau))) \xi d\xi.
\end{aligned} \tag{133}$$

By replacing t by $(t - \tau)$, η by λ and m by 1 in the precedent calculation, we have:

$$\int_0^{+\infty} G(x, t, \xi, \tau) \varphi_1(\xi) d\xi = \varphi_1(x), \quad x > 0. \quad (134)$$

2. By the definitions of the functions G and $\Phi = \varphi_2$ given in (35) and (42), respectively, we have:

$$\int_0^{+\infty} G(x, t, \xi, \tau) \varphi_2(\xi) d\xi = -\frac{\mu}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} (\exp(-(x-\xi)^2/4(t-\tau)) - \exp(-(x+\xi)^2/4(t-\tau))) \sinh(\lambda\xi) d\xi. \quad (135)$$

We first compute $\int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sinh(\lambda\xi) d\xi$.

By doing the change of variables $\zeta = (x - \xi)/2\sqrt{t - \tau}$, we have:

$$\begin{aligned} \int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \exp(\lambda\xi) d\xi = \\ 2\sqrt{t-\tau} \exp(\lambda x) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-\zeta^2 - 2\lambda\sqrt{t-\tau}\zeta) d\zeta. \end{aligned} \quad (136)$$

By writing:

$$\zeta^2 + 2\lambda\sqrt{t-\tau}\zeta = (\zeta + \lambda\sqrt{t-\tau})^2 - \lambda^2(t-\tau) \quad (137)$$

and doing the change of variables $\sigma = \zeta + \lambda\sqrt{t-\tau}$, we have:

$$\begin{aligned} 2\sqrt{t-\tau} \exp(\lambda x) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-\zeta^2 - 2\lambda\sqrt{t-\tau}\zeta) d\zeta = \\ = 2\sqrt{t-\tau} \exp(\lambda x + \lambda^2(t-\tau)) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-(\zeta + \lambda\sqrt{t-\tau})^2) d\zeta \\ = 2\sqrt{t-\tau} \exp(\lambda x + \lambda^2(t-\tau)) \int_{-\infty}^{x/2\sqrt{t-\tau} + \lambda\sqrt{t-\tau}} \exp(-\sigma^2) d\sigma \\ = \sqrt{\pi(t-\tau)} \exp(\lambda x + \lambda^2(t-\tau)) \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right), \end{aligned} \quad (138)$$

where erf is the error function, defined by:

$$\operatorname{erf}(z) = \int_0^z \exp(-\xi^2) d\xi, \quad z \in \mathbb{R}. \quad (139)$$

Hence, we have:

$$\begin{aligned} \int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \exp(\lambda\xi) d\xi = \\ \sqrt{\pi(t-\tau)} \exp(\lambda x + \lambda^2(t-\tau)) \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right). \end{aligned} \quad (140)$$

By replacing λ by $-\lambda$ in the previous calculations, we have:

$$\int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \exp(-\lambda\xi) d\xi = \sqrt{\pi(t-\tau)} \exp(-\lambda x + \lambda^2(t-\tau)) \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right). \quad (141)$$

Therefore, we have:

$$\begin{aligned} & \int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sinh(\lambda\xi) d\xi = \\ &= \frac{\sqrt{\pi(t-\tau)}}{2} \exp(\lambda^2(t-\tau)) \left(\exp(\lambda x) \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right) - \right. \\ & \left. \exp(-\lambda x) \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right) \right). \end{aligned} \quad (142)$$

By similar calculations, we have:

$$\begin{aligned} & \int_0^{+\infty} \exp(-(x+\xi)^2/4(t-\tau)) \sinh(\lambda\xi) d\xi = \\ &= \frac{\sqrt{\pi(t-\tau)}}{2} \exp(\lambda^2(t-\tau)) \left(\exp(-\lambda x) \left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right) - \right. \\ & \left. \exp(\lambda x) \left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right) \right). \end{aligned} \quad (143)$$

Then, we have:

$$\int_0^{+\infty} G(x, t, \xi, \tau) \varphi_2(\xi) d\xi = \exp(\lambda^2(t-\tau)) \varphi_2(x), \quad x > 0. \quad (144)$$

3. By the definitions of the functions G and $\Phi = \varphi_3$ given in (35) and (42), respectively, we have:

$$\begin{aligned} \int_0^{+\infty} G(x, t, \xi, \tau) \varphi_3(\xi) d\xi &= -\frac{\mu}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} \left(\exp(-(x-\xi)^2/4(t-\tau)) - \right. \\ & \left. \exp(-(x+\xi)^2/4(t-\tau)) \right) \sin(\lambda\xi) d\xi. \end{aligned} \quad (145)$$

We first compute $\int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sin(\lambda\xi) d\xi$.

By doing the change of variables $\zeta = (x-\xi)/2\sqrt{t-\tau}$, we have:

$$\begin{aligned} & \int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sin(\lambda\xi) d\xi = \\ & 2\sqrt{t-\tau} \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-\zeta^2) \left(\sin(\lambda x) \cos(2\lambda\sqrt{t-\tau}\zeta) - \cos(\lambda x) \sin(2\lambda\sqrt{t-\tau}\zeta) \right) d\zeta. \end{aligned} \quad (146)$$

By using the identities (see [20], p. 4):

$$\int \exp(-\zeta^2) \cos(\alpha\zeta) d\zeta = \frac{\sqrt{\pi}}{4} \exp(-\alpha^2/4) \left(\operatorname{erf}\left(\zeta + \frac{\alpha}{2}i\right) + \operatorname{erf}\left(\zeta - \frac{\alpha}{2}i\right) \right) \quad (147)$$

and

$$\int \exp(-\zeta^2) \sin(\alpha\zeta) d\zeta = \frac{\sqrt{\pi}i}{4} \exp(-\alpha^2/4) \left(\operatorname{erf}\left(\zeta + \frac{\alpha}{2}i\right) - \operatorname{erf}\left(\zeta - \frac{\alpha}{2}i\right) \right), \quad (148)$$

where $\alpha \in \mathbb{R}$ and i denotes the imaginary unit, we have:

$$\begin{aligned} & \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-\zeta^2) \cos(2\lambda\sqrt{t-\tau}\zeta) d\zeta = \\ & \frac{\sqrt{\pi}}{4} \exp(-\lambda^2(t-\tau)) \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) + 2 \right) \end{aligned} \quad (149)$$

and

$$\begin{aligned} & \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp(-\zeta^2) \sin(2\lambda\sqrt{t-\tau}\zeta) d\zeta = \\ & \frac{\sqrt{\pi}i}{4} \exp(-\lambda^2(t-\tau)) \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) \right). \end{aligned} \quad (150)$$

Then, we have:

$$\begin{aligned} & \int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sin(\lambda\xi) d\xi = \\ & \frac{\sqrt{\pi(t-\tau)}}{2} \exp(-\lambda^2(t-\tau)) \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) \sin(\lambda x) + \right. \\ & \left. \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) \sin(\lambda x) - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) i \cos(\lambda x) + \right. \\ & \left. \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) i \cos(\lambda x) + 2 \sin(\lambda x) \right). \end{aligned} \quad (151)$$

By similar calculations, we have:

$$\begin{aligned} & \int_0^{+\infty} \exp(-(x+\xi)^2/4(t-\tau)) \sin(\lambda\xi) d\xi = \\ & \frac{\sqrt{\pi(t-\tau)}}{2} \exp(-\lambda^2(t-\tau)) \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) i \cos(\lambda x) - \right. \\ & \left. \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) i \cos(\lambda x) + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) \sin(\lambda x) + \right. \\ & \left. \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) \sin(\lambda x) - 2 \sin(\lambda x) \right). \end{aligned} \quad (152)$$

Then, we have:

$$\int_0^{+\infty} G(x, t, \xi, \tau) \varphi_3(\xi) d\xi = \exp(-\lambda^2(t-\tau)) \varphi_3(x), \quad x > 0. \quad (153)$$

The proofs of propositions 2.5, 2.6 and 2.7 follow from the expression for u given in (34), the expression for $\int_0^{+\infty} G(x, t, \xi, 0)h(\xi)d\xi$ obtained in (132) and the expression for $\int_0^{+\infty} G(x, t, \xi, \tau)\Phi(\xi)d\xi$ obtained in (134), (144) and (153), respectively.